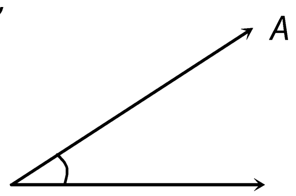
1. TRIGONOMETRY

The word trigonometry is originated from the greek word 'tri' means three, 'gonia' means angle and "metron" means measure. Hence the word trigonometry means three angle measure i.e. it is the study of geometrical figures which have three angles i.e. triangle.

1.1 ANGLE

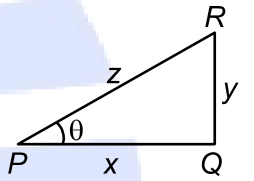
A measure formed between two rays having a common initial point is called an angle. The two rays are called the arms or sides of the angle and the common initial point is called the vertex of the angle.



In the above figure *OA* is said to be 'initial side' and the other ray is said to be 'terminal side'.

1.2 TRIGONOMETRIC RATIOS (T-RATIOS)

Let PQR be a right angle triangle with $\angle PQR = 90°$. Let $\angle QPR$ be θ. Notice that 0°< θ < 90° *i.e,* θ is an acute angle.



In right-angled Δ*PQR*, let base = *PQ* = *x* units,
Perpendicular = *QR* = *y* units and hypotenuse = *PR* = *z* units.

Trigonometric ratios for θ are defined as below:

(i) sine θ = $\frac{\text{perpendicular}}{\text{hypotenuse}} = \frac{y}{z}$, and is written as sin θ.

(ii) cosine θ = $\frac{\text{base}}{\text{hypotenuse}} = \frac{x}{z}$, and is written as cos θ.

(iii) tangent θ = $\frac{\text{perpendicular}}{\text{base}} = \frac{y}{x}$, and is written as tan θ.

(iv) cosecant θ = $\frac{\text{hypotenuse}}{\text{perpendicular}} = \frac{z}{y}$, and is written as cosec θ.

(v) secant θ = $\frac{\text{hypotenuse}}{\text{base}} = \frac{z}{x}$, and is written as sec θ.

(vi) cotangent θ = $\frac{\text{base}}{\text{perpendicular}} = \frac{x}{y}$, and is written as cot θ.

1.3 RECIPROCAL RELATION

We have

(i) cosec θ = $\frac{1}{\sin\theta}$ (ii) sec θ = $\frac{1}{\cos\theta}$ (iii) cot θ = $\frac{1}{\tan\theta}$

1.4 POWER OF T-RATIOS

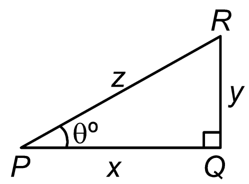
We write $(\sin\theta)^2 = \sin^2\theta$; $(\sin\theta)^3 = \sin^3\theta$; $(\cos\theta)^3 = \cos^3\theta$; etc.

1.5 QUOTIENT RELATION OF T-RATIOS

Theorem 1: For any acute angle θ, prove that

(i) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$; **(ii)** $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$;

Proof : Consider a right-angled Δ*PQR* in which ∠*Q* = 90° and ∠*P* = θ°. Let *PQ* = *x* units, *RQ* = *y* units and *PR* = *z* units. Then,



(i) $\tan\theta = \dfrac{y}{x} = \dfrac{(y/z)}{(x/z)}$ [dividing num. and denom. by z]

$= \dfrac{\sin\theta}{\cos\theta}$

∴ $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

(ii) $\cot\theta = \dfrac{x}{y} = \dfrac{(x/z)}{(y/z)}$ [dividing num. and denom. by z]

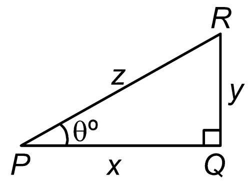
$= \dfrac{\cos\theta}{\sin\theta}$

1.6 SQUARE RELATION

Theorem 2: For any acute angle θ, prove that

(i) $\sin^2\theta + \cos^2\theta = 1$; **(ii)** $1 + \tan^2\theta = \sec^2\theta$; **(iii)** $1 + \cot^2\theta = \operatorname{cosec}^2\theta$.

Proof: Consider a right-angle Δ*PQR* in which ∠*Q* = 90° and ∠*P* = θ°. Let *PQ* = *x* units, *RQ* = *y* units and *PR* = *z* units.



Then, by Pythagoras' theorem, we have

$$x^2 + y^2 = z^2$$

Now, (i) $\sin^2\theta + \cos^2\theta = \left(\dfrac{y}{z}\right)^2 + \left(\dfrac{x}{z}\right)^2 = \left(\dfrac{y^2}{z^2} + \dfrac{x^2}{z^2}\right) = \dfrac{(x^2 + y^2)}{z^2} = \dfrac{z^2}{z^2} = 1$

[∵ $x^2 + y^2 = z^2$]

∴ $\sin^2\theta + \cos^2\theta = 1$

(ii) $$1+\tan^2\theta = 1+\left(\frac{y}{x}\right)^2 = 1+\frac{y^2}{x^2} = \frac{y^2+x^2}{x^2} = \frac{z^2}{x^2}$$ [∵ $x^2+y^2=z^2$]

$$= \left(\frac{z}{x}\right)^2 = \sec^2\theta$$

∴ $$1+\tan^2\theta = \sec^2\theta$$

(iii) $$1+\cot^2\theta = 1+\left(\frac{x}{y}\right)^2 = 1+\frac{x^2}{y^2} = \frac{x^2+y^2}{y^2} = \frac{z^2}{y^2}$$ [∵ $x^2+y^2=z^2$]

$$= \left(\frac{z}{y}\right)^2 = \text{cosec}^2\theta$$

∴ $$1+\cot^2\theta = \text{cosec}^2\theta.$$

Note : Some Pythagorean triplets are :
(i) 3, 4, 5 (ii) 5, 12, 13 (iii) 8, 15, 17 (iv) 7, 24, 25 (v) 9, 40, 41

2. TRIGONOMETRIC RATIOS OF SOME SPECIFIC ANGLES

2.1 TRIGONOMETRIC RATIOS OF 45º

Let ΔABC be a right-angled triangle in which $\angle B$ = 90º and $\angle A$ = 45º.

Then, $\angle C$ = 45º [Angle sum property]

$\angle A = \angle C \Rightarrow AB = BC$. [Side opposite to equal angles]

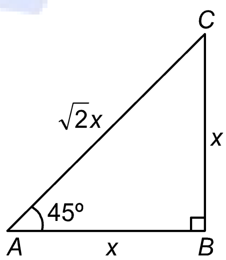
Let $AB = BC = x$ units. Then,

$AC = \sqrt{AB^2+BC^2} = \sqrt{x^2+x^2} = \sqrt{2x^2} = \sqrt{2}x$ units.

∴ base = AB = x units;

perpendicular = BC = x units and

hypotenuse = AC = $\sqrt{2}x$ units.



∴ $$\sin 45^\circ = \frac{BC}{AC} = \frac{x}{\sqrt{2}x} = \frac{1}{\sqrt{2}};$$

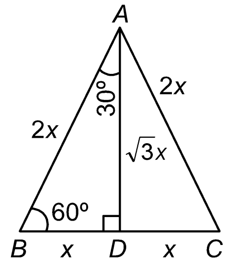
$$\cos 45^\circ = \frac{AB}{AC} = \frac{x}{\sqrt{2}x} = \frac{1}{\sqrt{2}};$$

$$\tan 45^\circ = \frac{BC}{AB} = \frac{x}{x} = 1;$$ $$\text{cosec } 45^\circ = \frac{1}{\sin 45^\circ} = \sqrt{2};$$

$$\sec 45^\circ = \frac{1}{\cos 45^\circ} = \sqrt{2};$$ $$\cot 45^\circ = \frac{1}{\tan 45^\circ} = 1.$$

2.2 TRIGONOMETRIC RATIOS OF 60º AND 30º

Consider an equilateral Δ*ABC* with each side equal to 2*x*.
Then, each angle of Δ*ABC* is 60º.
From A, draw $AD \perp BC$.
Then, clearly, *BD* = *DC* = *x*.
Also, $\angle ADB$ = 90º.
∴ $\angle BAD$ = 30º. [Angle sum property]
From right-angled Δ*ADB*, we have:



$$AD = \sqrt{AB^2 - BD^2} = \sqrt{(2x)^2 - x^2} = \sqrt{4x^2 - x^2} = \sqrt{3x^2} = \sqrt{3}x$$

2.2.1 T-Ratios of 60º

In right-angled Δ*ADB*, we have

base = *BD* = *x*; perpendicular = *AD* = $\sqrt{3}x$ and hypotenuse = *AB* = 2*x*

∴

$$\sin 60^\circ = \frac{AD}{AB} = \frac{\sqrt{3}x}{2x} = \frac{\sqrt{3}}{2}; \qquad \cos 60^\circ = \frac{BD}{AB} = \frac{x}{2x} = \frac{1}{2};$$

$$\tan 60^\circ = \frac{AD}{BD} = \frac{\sqrt{3}x}{x} = \sqrt{3}; \qquad \text{cosec } 60^\circ = \frac{1}{\sin 60^\circ} = \frac{2}{\sqrt{3}};$$

$$\sec 60^\circ = \frac{1}{\cos 60^\circ} = 2; \qquad \cot 60^\circ = \frac{1}{\tan 60^\circ} = \frac{1}{\sqrt{3}}.$$

2.2.2 T-Ratios of 30º

In right-angled Δ*ADB*, we have

base = *AD* = $\sqrt{3}x$, perpendicular = *BD* = *x* and hypotenuse = *AB* = 2*x*.

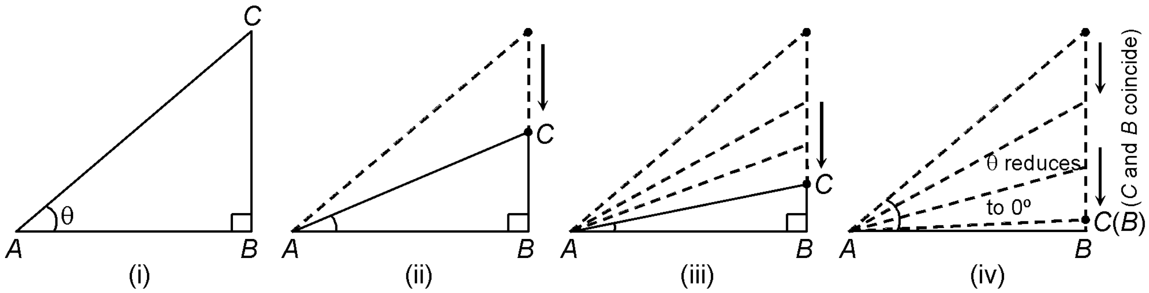
∴

$$\sin 30^\circ = \frac{BD}{AB} = \frac{x}{2x} = \frac{1}{2}; \qquad \cos 30^\circ = \frac{AD}{AB} = \frac{\sqrt{3}x}{2x} = \frac{\sqrt{3}}{2};$$

$$\tan 30^\circ = \frac{BD}{AD} = \frac{x}{\sqrt{3}x} = \frac{1}{\sqrt{3}}; \qquad \text{cosec } 30^\circ = \frac{1}{\sin 30^\circ} = 2;$$

$$\sec 30^\circ = \frac{1}{\cos 30^\circ} = \frac{2}{\sqrt{3}}; \qquad \cot 30^\circ = \frac{1}{\tan 30^\circ} = \sqrt{3};$$

2.3 TRIGONOMETRIC RATIOS OF 0º



In the figure (i), ΔABC is right angled at B and ∠BAC = θ.

In figure (ii) ∠BAC is reduced and it is less than θ. Here, we observe that the point C moves closer to the point B.

In figure (iii), ∠BAC is very small and the point C is also very close to the point B.

In figure (iv), ∠BAC just reduces to 0º and the point C coincides with the point B, i.e., BC = 0 and AB = AC. Then by definition, the values of the trigonometric ratios of 0º are as under:

$$\sin 0^\circ = \frac{BC}{AC} = \frac{0}{AC} = 0$$

$$\cos 0^\circ = \frac{AB}{AC} = \frac{AC}{AC} = 1$$

$$\tan 0^\circ = \frac{\sin 0^\circ}{\cos 0^\circ} = \frac{0}{1} = 0$$

$$\text{cosec } 0^\circ = \frac{1}{\sin 0^\circ} = \frac{1}{0}$$

$$\sec 0^\circ = \frac{1}{\cos 0^\circ} = \frac{1}{1} = 1$$

$$\cot 0^\circ = \frac{1}{\tan 0^\circ} = \frac{1}{0}$$

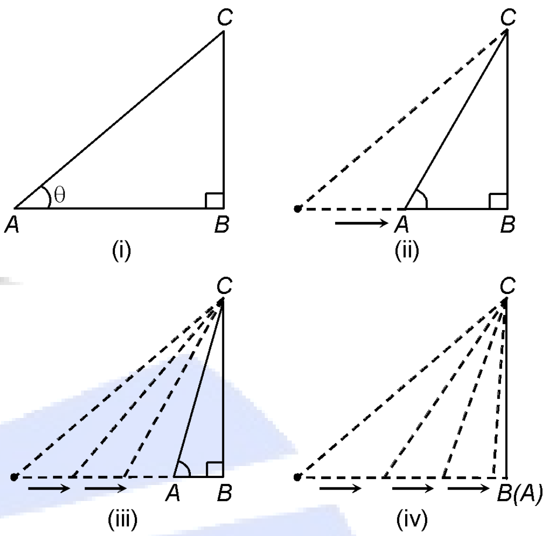
Hence, we have

sin 0º = 0, cos 0º = 1, tan 0º = 0, sec 0º = 1

The values of cosec 0º and cot 0º are not defined as real numbers.

2.4 TRIGONOMETRIC RATIOS OF 90º

We observe from the figures (i, ii, iii, iv) that as the point A moves closer to the point B, the angle $\angle BAC$ becomes larger and larger and ultimately when A coincides with B, the angle $\angle BAC$ becomes equal to 90º.



Thus, when $\angle BAC$ = 90º, we have AB = 0

$BC = AC$ because AC and BC coincide.

Now, we get

$$\sin 90^\circ = \frac{BC}{AC} = \frac{AC}{AC} = 1$$

$$\cos 90^\circ = \frac{AB}{AC} = \frac{0}{AC} = 0$$

$$\tan 90^\circ = \frac{\sin 90^\circ}{\cos 90^\circ} = \frac{1}{0}$$

$$\text{cosec } 90^\circ = \frac{1}{\sin 90^\circ} = \frac{1}{1} = 1 \quad \Rightarrow \quad \sec 90^\circ = \frac{1}{\cos 90^\circ} = \frac{1}{0} \quad \Rightarrow \quad \cot 90^\circ = \frac{\cos 90^\circ}{\sin 90^\circ} = \frac{0}{1} = 0$$

Hence, we have the values of the trigonometric ratios of 90º as under:

sin 90º = 1, cos 90º = 0, cosec 90º = 1, cot 90º = 0

The values of sec 90º and tan 90º are not defined as real numbers.

2.5 VALUES OF ALL THE TRIGONOMETRIC RATIOS OF 0°, 30°, 45°, 60° AND 90°.

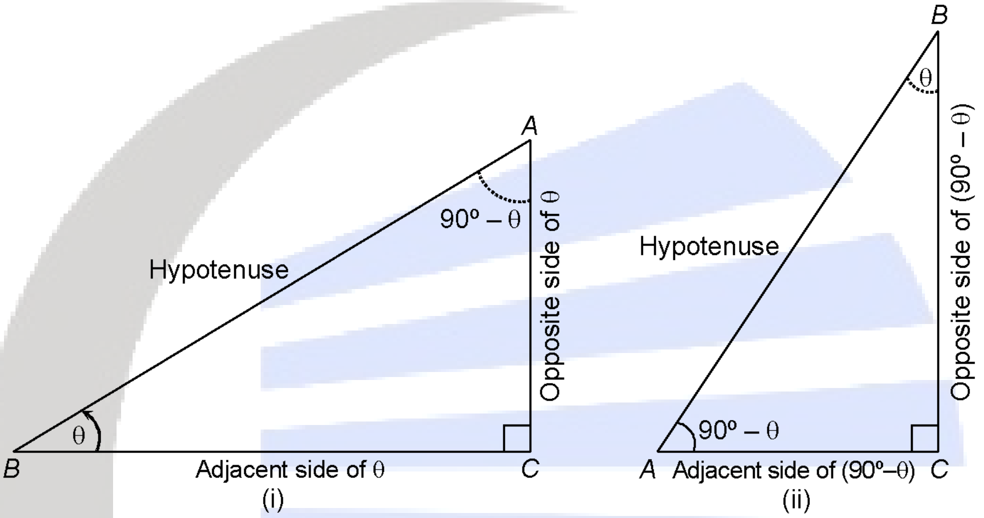
θ	**0°**	**30°**	**45°**	**60°**	**90°**
sin θ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan θ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	n. d.
cosecθ	n. d.	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
sec θ	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	n. d.

cot θ	n. d.	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

3. TRIGONOMETRIC RATIOS OF COMPLEMENTARY ANGLES

Let us construct a right-angled ΔABC in which $\angle ABC = \theta$, $\angle BCA = 90^\circ$.

Then we have $\angle BAC = (90^\circ - \theta)$. Angles θ and $(90^\circ - \theta)$ are complementary angles.



In figure (i), the sides are labelled corresponding to angle θ and in figure (ii), the sides are labelled corresponding to angle $(90^\circ - \theta)$.

From figure (ii), we have $\sin(90^\circ - \theta) = \frac{BC}{BA}$ and from figure (i), we have $\frac{BC}{BA} = \cos\theta$

∴ $\sin(90^\circ - \theta) = \frac{BC}{BA} = \cos\theta$

Similarly, $\cos(90^\circ - \theta) = \frac{AC}{BA} = \sin\theta$

$\tan(90^\circ - \theta) = \frac{BC}{AC} = \cot\theta$

$\text{cosec}(90^\circ - \theta) = \frac{BA}{BC} = \sec\theta$

$\sec(90^\circ - \theta) = \frac{BA}{AC} = \text{cosec}\,\theta$

$\cot(90^\circ - \theta) = \frac{AC}{BC} = \tan\theta$

If *A* and *B* are two complementary acute angles, i.e., $A + B = 90^\circ$, then we have

$$\sin A = \sin (90^\circ - B) = \cos B$$
$$\cos A = \cos (90^\circ - B) = \sin B$$
$$\tan A = \tan (90^\circ - B) = \cot B$$
$$\operatorname{cosec} A = \operatorname{cosec} (90^\circ - B) = \sec B$$
$$\sec A = \sec (90^\circ - B) = \operatorname{cosec} B$$
$$\cot A = \cot (90^\circ - B) = \tan B$$

4. TRIGONOMETRIC IDENTITIES

An equation involving trigonometric ratios of an angle is called a trigonometric identity, if it is true for all values of the angle(s) involved.

Following are the three trigonometric identities which are used to solve the basic trigonometric equations.

(i) $\sin^2\theta + \cos^2\theta = 1$; (ii) $1 + \tan^2\theta = \sec^2\theta$;

(iii) $1 + \cot^2\theta = \cos ec^2\theta$.