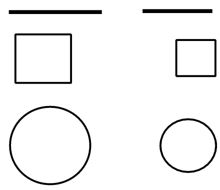
1. SIMILAR TRIANGLES

Similar figures: Geometric figures which have the same shape but different sizes are known as similar figures.

Illustrations:

1. Any two line-segments are similar
2. Any two squares are similar
3. Any two circles are similar



Two congruent figures are always similar but two similar figures need not be congruent.

Similar polygons: Two polygons of the same number of sides are said to be similar if

(i) their corresponding angles are equal (i.e., they are equiangular) and

(ii) their corresponding sides are in the same ratio (or proportion)

Similar triangles: Since triangles are also polygons, the same conditions of similarity are applicable to them.

Two triangles are said to be similar if

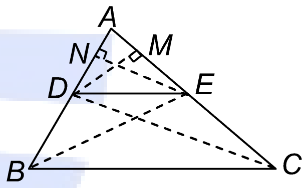
(i) their corresponding angles are equal and

(ii) their corresponding sides are in the same ratio (or proportion).

1.1 BASIC-PROPORTIONALITY THEOREM (Thales theorem)

Theorem 1: If a line is drawn parallel to one side of a triangle to intersect the other two sides in distinct points, the other two sides are divided in the same ratio.

Given: A triangle ABC in which a line parallel to side BC intersects other two sides AB and AC at D and E respectively



To prove: $\frac{AD}{DB} = \frac{AE}{EC}$.

Construction: Join BE and CD and draw $DM \perp AC$ and $EN \perp AB$.

Proof: area of $\Delta\ ADE$ $(= \frac{1}{2}\text{base} \times \text{height}) = \frac{1}{2} AD \times EN$.

(Taking AD as base)

So, $\text{ar}(ADE) = \frac{1}{2} AD \times EN$ [The area of $\Delta\ ADE$ is denoted as ar (ADE)]

Similarly, $\text{ar}(BDE) = \frac{1}{2} DB \times EN$,

$\text{ar}(ADE) = \frac{1}{2} AE \times DM$ and $\text{ar}(DEC) = \frac{1}{2} EC \times DM$. (Taking AE as base)

Therefore, $\frac{\text{ar}(ADE)}{\text{ar}(BDE)} = \frac{\frac{1}{2}AD \times EN}{\frac{1}{2}DB \times EN} = \frac{AD}{DB}$ …(i)

and $\frac{\text{ar}(ADE)}{\text{ar}(DEC)} = \frac{\frac{1}{2}AE \times DM}{\frac{1}{2}EC \times DM} = \frac{AE}{EC}$ …(ii)

$\text{ar}(BDE) = \text{ar}(DEC)$ ... (iii)

[Δ *BDE* and *DEC* are on the same base *DE* and between the same parallels *BC* and *DE*.]

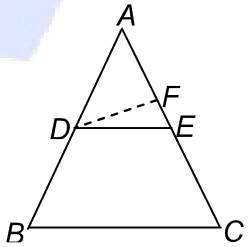
Therefore, from (i), (ii) and (iii), we have: $\frac{AD}{DB} = \frac{AE}{EC}$

Corollary: From above equation we have $\frac{DB}{AD} = \frac{EC}{AE}$

Adding '1' to both sides we have $\frac{DB}{AD} + 1 = \frac{EC}{AE} + 1$

$\frac{DB + AD}{AD} = \frac{EC + AE}{AE} \Rightarrow \frac{AB}{AD} = \frac{AC}{AE}$

Theorem 2: (Converse of BPT theorem) If a line divides any two sides of a triangle in the same ratio, prove that it is parallel to the third side.



Given: In Δ*ABC*, *DE* is a straight line such that $\frac{AD}{DB} = \frac{AE}{EC}$

To prove: *DE* || *BC*

Construction: If *DE* is not parallel to *BC*, draw *DF* meeting AC at *F*

Proof: In Δ*ABC*, let *DF* || *BC*

∴ $\frac{AD}{DB} = \frac{AF}{FC}$ …(i)

[∵ A line drawn parallel to one side of a Δ divides the other two sides in the same ratio.]

But $\frac{AD}{DB} = \frac{AE}{EC}$ …(ii) [given]

From (i) and (ii), we get

$$\frac{AF}{FC} = \frac{AE}{EC}$$

Adding 1 to both sides, we get $\frac{AF}{FC} + 1 = \frac{AE}{EC} + 1$

$\Rightarrow \quad \frac{AF + FC}{FC} = \frac{AE + EC}{EC}$

$\Rightarrow \quad \frac{AC}{FC} = \frac{AC}{EC} \quad \Rightarrow \quad FC = EC$

It is possible only when *E* and *F* coincide
Hence, $DE \parallel BC$.

2. CRITERIA FOR SIMILARITY OF TWO TRIANGLES

Two triangles are said to be similar if (i) their corresponding angles are equal and (ii) their corresponding sides are in the same ratio (or proportional).

Thus, two triangles *ABC* and $A'B'C'$ are similar if

(i) $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$ and

(ii) $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'}$

In this section, we shall make use of the theorems discussed in earlier sections to derive some criteria for similar triangles which in turn will imply that either of the above two conditions can be used to define the similarity of two triangles.

2.1 CHARACTERISTIC PROPERTY 1 (AAA SIMILARITY)

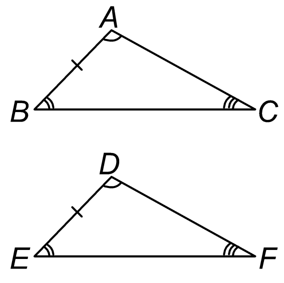
Theorem 3: If in two triangles, the corresponding angles are equal, then the triangles are similar.

Given: Two triangles *ABC* and *DEF* in which $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$.

To prove: $\Delta ABC \sim \Delta DEF$

Proof:

Case 1: When *AB* = *DE*



In triangles *ABC* and *DEF*, we have

$\angle A = \angle D$ [Given]

$AB = DE$ [Given]

$\angle B = \angle E$ [Given]

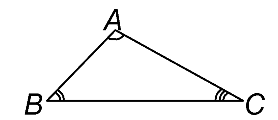
$\therefore \quad \Delta ABC \cong \Delta DEF$ [By ASA congruency]

$\Rightarrow \quad BC = EF$ and $AC = DF$ [c.p.c.t.]

Thus, $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$ [corresponding sides of similar Δs are proportional]

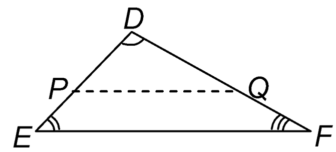
Hence, $\Delta ABC \sim \Delta DEF$

Case 2: When $AB < DE$



Let P and Q be points on DE and DF respectively such that $DP = AB$ and $DQ = AC$. Join PQ.

In ΔABC and ΔDPQ, we have



$AB = DP$	[By construction]
$\angle A = \angle D$	[Given]
$AC = DQ$	[By construction]

∴	$\Delta ABC \cong \Delta DPQ$	[By SAS congruency]	
∴	$\angle ABC = \angle DPQ$	[c.p.c.t.]	…(i)
But	$\angle ABC = \angle DEF$	[Given]	…(ii)
∴	$\angle DPQ = \angle DEF$	[c.p.c.t.]	

But $\angle DPQ$ and $\angle DEF$ are corresponding angles.

⇒ $PQ \parallel EF$

∴ $\dfrac{DP}{DE} = \dfrac{DQ}{DF}$ [Corollary to BPT Theorem]

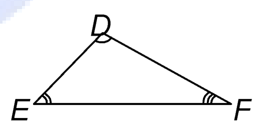
∴ $\dfrac{AB}{DE} = \dfrac{AC}{DF}$ [∵ $DP = AB$ and $DQ = AC$ (by construction)]

Similarly $\dfrac{AB}{DE} = \dfrac{BC}{EF}$

∴ $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{AC}{DF}$

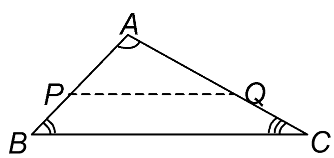
Hence, $\Delta ABC \sim \Delta DEF$.

Case 3: When $AB > DE$



Let P and Q be points on AB and AC respectively such that $AP = DE$ and $AQ = DF$. Join PQ.

In ΔAPQ and ΔDEF, we have



$AP = DE$	[By construction]
$AQ = DF$	[By construction]
$\angle A = \angle D$	[Given]

∴	$\Delta APQ \cong \Delta DEF$	[By SAS congruency]	
∴	$\angle APQ = \angle DEF$	[c.p.c.t.]	…(i)
But	$\angle DEF = \angle ABC$	[Given]	…(ii)

From (i) and (ii) we have

∴ $\angle APQ = \angle ABC$

But $\angle APQ$ and $\angle ABC$ are corresponding angles

∴ $PQ \parallel BC$

∴ $\frac{AP}{AB} = \frac{AQ}{AC}$ [Corollary to BPT Theorem]

∴ $\frac{DE}{AB} = \frac{DF}{AC}$ [∵ $AP = DE$ and $AQ = DF$ (by construction)]

Similarly, $\frac{DE}{AB} = \frac{EF}{BC}$

Thus, $\frac{DE}{AB} = \frac{EF}{BC} = \frac{DF}{AC}$

or $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$

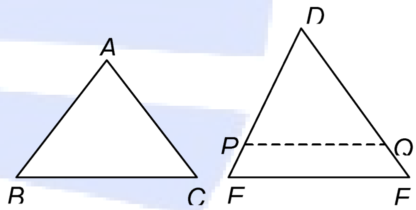
Hence, $\Delta ABC \sim \Delta DEF$.

2.2 CHARACTERISTIC PROPERTY 2 (SSS SIMILARITY)

Theorem 4: If the corresponding sides of two triangles are proportional, then they are similar.

Given: Two triangles ABC and DEF such that

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$



To prove: $\Delta ABC \sim \Delta DEF$

Construction: Let P and Q be points on DE and DF respectively such that DP = AB and DQ = AC. Join PQ

Proof: $\frac{AB}{DE} = \frac{AC}{DF}$ [Given]

⇒ $\frac{DP}{DE} = \frac{DQ}{DF}$ …(i)

[∵ $DP = AB$ and $DQ = AC$ (by construction)]

In ΔDEF, we have

$\frac{DP}{DE} = \frac{DQ}{DF}$ [From (i)]

∴ PQ || EF [By the converse of BPT]

∴ $\angle DPQ = \angle DEF$ and $\angle DQP = \angle DFE$ [Corresponding angles]

∴ $\Delta DPQ = \sim \Delta DEF$ [By AA similarity] …(ii)

∴ $\frac{DP}{DE} = \frac{PQ}{EF}$

or $\frac{AB}{DE} = \frac{PQ}{EF}$ [∵ $DP = AB$] …(iii)

But $\frac{AB}{DE} = \frac{BC}{EF}$ [given] …(iv)

From equations (iii) and (iv), we have $\frac{PQ}{EF} = \frac{BC}{EF}$

⇒ $BC = PQ$ …(v)

In ΔABC and ΔDPQ, we have

$AB = DP$ [By construction]

$AC = DQ$ [By construction]

$BC = PQ$ [By (v)]

∴ $\Delta ABC \cong \Delta DPQ$ [by SSS congruency]

⇒ $\Delta ABC \sim \Delta DPQ$ [∵ $\Delta ABC \cong DPQ \Leftrightarrow \Delta ABC \sim \Delta DPQ$] ... (vi)

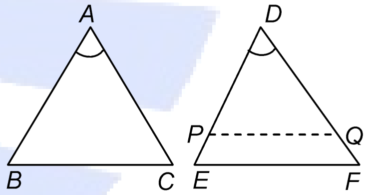
From equation (ii) and (vi), we get

$\Delta ABC \sim \Delta DEF$

2.3 CHARACTERISTIC PROPERTY 3 (SAS SIMILARITY)

Theorem 5: If one angle of a triangle is equal to one angle of the other and the sides including these angles are proportional than the two triangles are similar.

Given: Two triangle ABC and DEF such that $\angle A = \angle D$ and $\frac{AB}{DE} = \frac{AC}{DF}$



To prove: $\Delta ABC \sim \Delta DEF$

Construction: Let P and Q be points on DE and DF respectively such that $DP = AB$ and $DQ = AC$. Join PQ.

Proof: In ΔABC and ΔDPQ, we have

$AB = DP$ [By construction]

$AC = DQ$ [By construction]

$\angle A = \angle D$ [Given]

∴ $\Delta ABC \cong \Delta DPQ$ [By SAS congruency] …(i)

Now, $\frac{AB}{DE} = \frac{AC}{DF}$ …(ii)

∴ $\frac{DP}{DE} = \frac{DQ}{DF}$ [∵ $AB = DP$ and $AC = DQ$ (by construction)]

In ΔDEF, we have

$$\frac{DP}{DE}=\frac{DQ}{DF}$$ [From (ii)]

∴ $PQ \parallel EF$ [By the converse of BPT]

∴ $\angle DPQ = \angle DEF$ and $\angle DQP = \angle DFE$ [Corresponding angles]

∴ $\Delta DPQ \sim \Delta DEF$ [By AA similarity] …(iii)

From equations (i) and (iii), we get $\Delta ABC \sim \Delta DEF$.

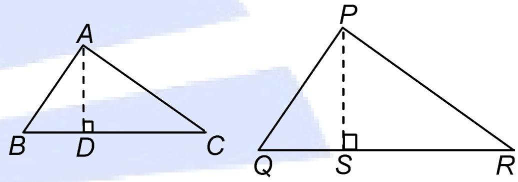
3. AREAS OF SIMILAR TRIANGLES

Theorem 6: The ratio of the areas of two similar triangles is equal to the ratio of the squares of their corresponding sides.

Given: ΔABC and ΔPQR such that $\Delta ABC \sim \Delta PQR$.

To prove: $$\frac{\text{ar}(\Delta ABC)}{\text{ar}(\Delta PQR)}=\frac{AB^2}{PQ^2}=\frac{BC^2}{QR^2}=\frac{CA^2}{RP^2}$$

Construction: Draw $AD \perp BC$ and $PS \perp QR$



Proof: $$\frac{\text{ar}(\Delta ABC)}{\text{ar}(\Delta PQR)}=\frac{\frac{1}{2}\times BC\times AD}{\frac{1}{2}\times QR\times PS}$$ [∵ Area of triangle $=\frac{1}{2}$ base × height]

⇒ $$\frac{\text{ar}(\Delta ABC)}{\text{ar}(\Delta PQR)}=\frac{BC}{QR}\times\frac{AD}{PS}$$ …(i)

In ΔADB and ΔPSQ

$\angle B = \angle Q$ [∵ $\Delta ABC \sim \Delta PQR$]

$\angle ADB = \angle PSQ$ [Both 90°]

∴ $\Delta ADB \sim \Delta PSQ$ [By *AA* similarity]

⇒ $$\frac{AD}{PS}=\frac{AB}{PQ}$$ …(ii)

[Corresponding sides of similar triangles are proportional]

But $$\frac{AB}{PQ}=\frac{BC}{QR}$$ [∵ $\Delta ABC \sim \Delta PQR$]

∴ $$\frac{AD}{PS}=\frac{BC}{QR}$$ [Using (ii)] …(iii)

From (i) and (iii), we have

$$\frac{\text{ar}(\Delta ABC)}{\text{ar}(\Delta PQR)}=\frac{BC}{QR}\times\frac{BC}{QR}=\frac{BC^2}{QR^2}$$ …(iv)

Since $\Delta ABC \sim \Delta PQR$

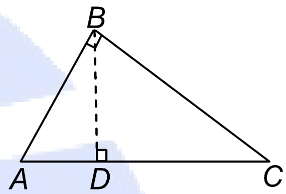
$\therefore \quad \frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP}$ ...(v)

Hence, $\frac{ar(\Delta ABC)}{ar(\Delta PQR)} = \frac{AB^2}{PQ^2} = \frac{BC^2}{QR^2} = \frac{CA^2}{RP^2}$ [From (iv) and (v)]

4. PYTHAGORAS THEOREM

Theorem 7: In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

Given: A right triangle *ABC*, right angled at *B*



To prove: $AC^2 = AB^2 + BC^2$

Construction: Draw $BD \perp AC$

Proof: In ΔADB and ΔABC, we have

$\angle ADB = \angle ABC$ [Each equal to 90°]

$\angle A = \angle A$ [Common]

$\therefore$ $\Delta ADB \sim \Delta ABC$ [By AA similarity]

$\Rightarrow \quad \frac{AD}{AB} = \frac{AB}{AC}$ [Corresponding sides of similar triangles are proportional]

$\Rightarrow \quad AB^2 = AD \times AC$ ...(i)

In ΔBCD and ΔACB, we have

$\angle CDB = \angle CBA$ [Each equal to 90°]

$\angle C = \angle C$ [Common]

By AA similarity criterion

$\Delta BCD \sim \Delta ACB$

$\therefore \quad \frac{BC}{AC} = \frac{DC}{BC}$

$\Rightarrow \quad BC^2 = DC \times AC$ ...(ii)

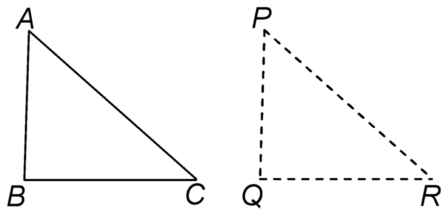
Adding equations (i) and (ii), we get

$$AB^2 + BC^2 = AD \times AC + DC \times AC$$

$$= AC(AD + DC) = AC \times AC = AC^2$$

Hence, $AC^2 = AB^2 + BC^2$.

Theorem 8: (Converse of Pythagoras theorem) In a triangle if the square of one side is equal to the sum of the squares of the other two sides, then the angle opposite to the first side is a right angle.



Given: A triangle ABC such that $AB^2 + BC^2 = AC^2$.

To prove: $\angle B = 90°$

Construction: Draw a ΔPQR right angled at Q such that $PQ = AB$ and $QR = BC$.

Proof: In right triangle PQR, we have

$PQ^2 + QR^2 = PR^2$ [By Pythagoras Theorem]

$\Rightarrow$ $AB^2 + BC^2 = PR^2$ …(i)

[$\because$ $PQ = AB$ and $QR = BC$ (by construction)]

But $AB^2 + BC^2 = AC^2$ [Given]…(ii)

From equation (i) and equation (ii), we get

$PR^2 = AC^2$

$\Rightarrow$ $PR = AC$ …(iii)

Now, in ΔABC and ΔPQR, we have

$AB = PQ$ and $BC = QR$ [By construction]

and $AC = PR$ [From (iii)]

$\therefore$ $\Delta ABC \cong \Delta PQR$ [By SSS congruency]

$\therefore$ $\angle B = \angle Q = 90°$ [c.p.c.t.]

Hence, $\angle B = 90°$.